

The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions



21st Annual

AMC 8

(American Mathematics Contest 8)

Solutions Pamphlet

Tuesday, NOVEMBER 15, 2005

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. *Dissemination at any time via copier, telephone, e-mail, World Wide Web or media of any type is a violation of the competition rules.*

Correspondence about the problems and solutions should be addressed to:

Ms. Bonnie Leitch, AMC 8 Chair / bleitch@earthlink.net
548 Hill Avenue, New Braunfels, TX 78130

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:

Attn: Publications
American Mathematics Competitions
University of Nebraska-Lincoln
P.O. Box 81606
Lincoln, NE 68501-1606

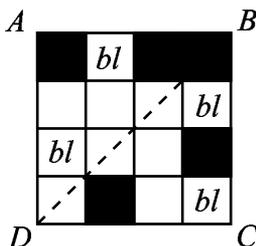
Copyright © 2005, The Mathematical Association of America

- (B) If multiplying a number by 2 results in 60, then the number must be 30. If 30 is divided by 2, the correct answer is 15.
- (C) Karl spent $5 \times \$2.50 = \12.50 on the folders. If he had purchased the folders a day later, he would have saved 20% of this total, or $0.20 \times \$12.50 = \2.50 .

OR

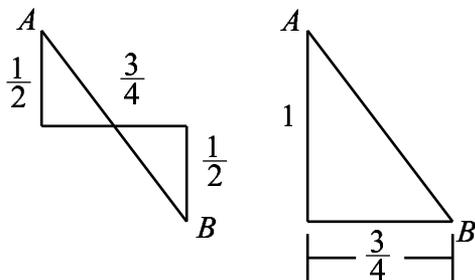
Karl could have bought five folders for the price of four in the 20%-off sale, so he could have saved \$2.50.

- (D) For diagonal BD to lie on a line of symmetry in square $ABCD$, the four small squares labeled bl must be colored black.



- (C) The perimeter of the triangle is $6.1 + 8.2 + 9.7 = 24$ cm. The perimeter of the square is also 24 cm. Each side of the square is $24 \div 4 = 6$ cm. The area of the square is $6^2 = 36$ square centimeters.
- (B) To get the minimum number of packs, purchase as many 24-packs as possible: three 24-packs contain 72 cans, which leaves $90 - 72 = 18$ cans. To get the remaining 18 cans, purchase one 12-pack and one 6-pack. The minimum number of packs is 5.
- (C) The number $2.00d5$ is greater than 2.005 for $d = 5, 6, 7, 8$ and 9 . Therefore, there are five digits satisfying the inequality.

7. (B) The diagram on the left shows the path of Bill's walk. As the diagram on the right illustrates, he could also have walked from A to B by first walking 1 mile south then $\frac{3}{4}$ mile east.



By the Pythagorean Theorem,

$$(AB)^2 = 1^2 + \left(\frac{3}{4}\right)^2 = 1 + \frac{9}{16} = \frac{25}{16},$$

so $AB = \frac{5}{4} = 1\frac{1}{4}$.

8. (E)

To check the possible answers, choose the easiest odd numbers for m and n . If $m = n = 1$, then

$$m + 3n = 4, \quad 3m - n = 2, \quad 3m^2 + 3n^2 = 6, \quad (mn + 3)^2 = 16 \text{ and } 3mn = 3.$$

This shows that (A), (B), (C) and (D) can be even when m and n are odd. On the other hand, because the product of odd integers is always odd, $3mn$ is always odd if m and n are odd.

Questions: Which of the expressions are always even if m and n are odd? What are the possibilities if m and n are both even? If one is even and the other odd?

9. (D) Triangle ACD is an isosceles triangle with a 60° angle, so it is also equilateral. Therefore, the length of \overline{AC} is 17.
10. (D) Covering the same distance three times as fast takes one-third the time. So Joe ran for 2 minutes. His total time was $6 + 2 = 8$ minutes.
11. (C) To add 6% sales tax to an item, multiply the price by 1.06. To calculate a 20% discount, multiply the price by 0.8. Because both actions require only multiplication, and because multiplication is commutative, the order of operations doesn't matter. Jack and Jill will get the same total.
- Note: Jack's final computation is $0.80(1.06 \times \$90.00)$ and Jill's is $1.06(0.80 \times \$90.00)$. Both yield the same product, \$76.32.

12. (D) You can solve this problem by guessing and checking. If Big Al had eaten 10 bananas on May 1, then he would have eaten $10 + 16 + 22 + 28 + 34 = 110$ bananas. This is 10 bananas too many, so he actually ate 2 fewer bananas each day. Thus, Big Al ate 8 bananas on May 1 and 32 bananas on May 5.

OR

The average number of bananas eaten per day was $\frac{100}{5} = 20$. Because the number of bananas eaten on consecutive days differs by 6 and there are an odd number of days, the average is also the median. Therefore, the average number of bananas he ate per day, 20, is equal to the number of bananas he ate on May 3. So on May 5 Big Al ate $20 + 12 = 32$ bananas.

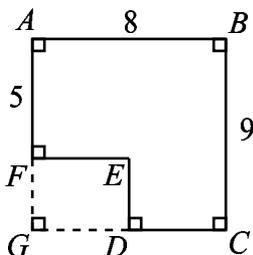
OR

Let x be the number of bananas that Big Al ate on May 5. The following chart documents his banana intake for the five days.

May 5	May 4	May 3	May 2	May 1
x	$x - 6$	$x - 12$	$x - 18$	$x - 24$

The total number of bananas Big Al ate was $5x - 60$, which must be 100. So Big Al ate $x = \frac{160}{5} = 32$ bananas on May 5.

13. (C)



Rectangle $ABCG$ has area $8 \times 9 = 72$, so rectangle $FEDG$ has area $72 - 52 = 20$. The length of \overline{FG} equals $DE = 9 - 5 = 4$, so the length of \overline{EF} is $\frac{20}{4} = 5$. Therefore, $DE + EF = 4 + 5 = 9$.

14. (B) Each team plays 10 games in its own division and 6 games against teams in the other division. So each of the 12 teams plays 16 conference games. Because each game involves two teams, there are $\frac{12 \times 16}{2} = 96$ games scheduled.
15. (C) Because the perimeter of such a triangle is 23, and the sum of the two equal side lengths is even, the length of the base is odd. Also, the length of the base is less than the sum of the other two side lengths, so it is less than half of 23. Thus the six possible triangles have side lengths 1, 11, 11; 3, 10, 10; 5, 9, 9; 7, 8, 8; 9, 7, 7 and 11, 6, 6.

16. **(D)** It is possible for the Martian to pull out at most 4 red, 4 white and 4 blue socks without having a matched set. The next sock it pulls out must be red, white or blue, which gives a matched set. So the Martian must select $4 \times 3 + 1 = 13$ socks to be guaranteed a matched set of five socks.
17. **(E)** Evelyn covered more distance in less time than Briana, Debra and Angela, so her average speed is greater than any of their average speeds. Evelyn went almost as far as Carla in less than half the time that it took Carla, so Evelyn's average speed is also greater than Carla's.

OR

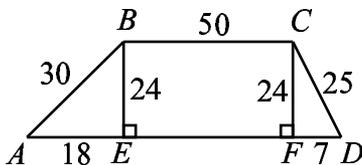
The ratio of distance to time, or average speed, is indicated by the slope of the line from the origin to each runner's point in the graph. Therefore, the line from the origin with the greatest slope will correspond to the runner with the greatest average speed. Because Evelyn's line has the greatest slope, she has the greatest average speed.

18. **(C)** The smallest three-digit number divisible by 13 is $13 \times 8 = 104$, so there are seven two-digit multiples of 13. The greatest three-digit number divisible by 13 is $13 \times 76 = 988$. Therefore, there are $76 - 7 = 69$ three-digit numbers divisible by 13.

OR

Because the integer part of $\frac{999}{13}$ is 76, there are 76 multiples of 13 less than or equal to 999. Because the integer part of $\frac{99}{13}$ is 7, there are 7 multiples of 13 less than or equal to 99. That means there are $76 - 7 = 69$ multiples of 13 between 100 and 999.

19. **(A)**



By the Pythagorean Theorem, $AE = \sqrt{30^2 - 24^2} = \sqrt{324} = 18$. (Or note that triangle AEB is similar to a 3-4-5 right triangle, so $AE = 3 \times 6 = 18$.)

Also $CF = 24$ and $FD = \sqrt{25^2 - 24^2} = \sqrt{49} = 7$. The perimeter of the trapezoid is $50 + 30 + 18 + 50 + 7 + 25 = 180$.

20. (A) Write the points where Alice and Bob will stop after each move and compare points.

Move	0	1	2	3	4	5	6
Alice:	12	5	10	3	8	1	6
Bob:	12	3	6	9	12	3	6

So Alice and Bob will be together again after six moves.

OR

If Bob does not move and Alice moves $9 + 5 = 14$ points or 2 points each time, they will still be in the same relative position from each other after each turn. If Bob does not move, they will be on the same point when Alice first stops on point 12, where she started. So Alice will have to move 2 steps 6 times to stop at her starting point.

21. (C) To make a triangle, select as vertices two dots from one row and one from the other row. To select two dots in the top row, decide which dot is not used. This can be done in three ways. There are also three ways to choose one dot to use from the bottom row. So there are $3 \times 3 = 9$ triangles with two vertices in the top row and one in the bottom. Similarly, there are nine triangles with one vertex in the top row and two in the bottom. This gives a total of $9 + 9 = 18$ triangles.

Note: Can you find the four noncongruent triangles?

22. (E) Neither the units of size nor the cost are important in this problem. So for convenience, suppose the small size costs \$1 and weighs 10 ounces. To determine the relative value, we compare the cost per unit weight.

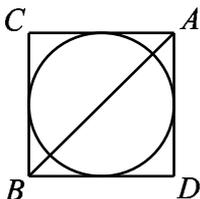
$$S : \frac{\$1.00}{10} = 10 \text{¢ per oz.}$$

$$M : \frac{\$1.50}{0.8 \times 20} = 9.375 \text{¢ per oz.}$$

$$L : \frac{1.3 \times \$1.50}{20} = 9.75 \text{¢ per oz.}$$

So the value, or buy, from best to worst is medium, large and small, that is MLS.

23. (B) Reflect the triangle and the semicircle across the hypotenuse \overline{AB} to obtain a circle inscribed in a square. The circle has area 4π . The radius of a circle with area 4π is 2. The side length of the square is 4 and the area of the square is 16. So the area of the triangle is 8.



24. **(B)** One way to solve the problem is to work backward, either dividing by 2 if the number is even or subtracting 1 if the number is odd.

$$200/2 \rightarrow 100/2 \rightarrow 50/2 \rightarrow 25 - 1 \rightarrow 24/2 \rightarrow 12/2 \rightarrow 6/2 \rightarrow 3 - 1 \rightarrow 2/2 \rightarrow 1$$

So if you press $[\times 2]$ $[+1]$ $[\times 2]$ $[\times 2]$ $[\times 2]$ $[+1]$ $[\times 2]$ $[\times 2]$ $[\times 2]$ or 9 keystrokes, you can reach “200” from “1.”

To see that no sequence of eight keystrokes works, begin by noting that of the four possible sequences of two keystrokes, $[\times 2]$ $[\times 2]$ produces the maximum result. Furthermore, $[+1]$ $[\times 2]$ produces a result larger than either $[\times 2]$ $[+1]$ or $[+1]$ $[+1]$. So the largest possible result of a sequence of eight keystrokes is “256,” produced by either

$$[\times 2] [\times 2]$$

or

$$[+1] [\times 2] [\times 2] [\times 2] [\times 2] [\times 2] [\times 2] [\times 2].$$

The second largest result is “192,” produced by

$$[\times 2] [+1] [\times 2] [\times 2] [\times 2] [\times 2] [\times 2] [\times 2].$$

Thus no sequence of eight keystrokes produces a result of “200.”

25. **(A)** Because the circle and square share the same interior region and the area of the two exterior regions indicated are equal, the square and the circle must have equal area. The area of the square is 2^2 or 4. Because the area of both the circle and the square is 4, $4 = \pi r^2$. Solving for r , the radius of the circle, yields $r^2 = \frac{4}{\pi}$, so $r = \sqrt{\frac{4}{\pi}} = \frac{2}{\sqrt{\pi}}$.

Note: It is not necessary that the circle and square have the same center.

The
AMERICAN MATHEMATICS COMPETITIONS
are Sponsored by

The Mathematical Association of America
University of Nebraska-Lincoln
The Akamai Foundation

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mu Alpha Theta
National Council of Teachers of Mathematics
National Assessment & Testing
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.