The Mathematical Association of America
American Mathematics Competitions

20th Annual

AMC 8

(American Mathematics Contest 8)

Solutions Pamphlet

Tuesday, NOVEMBER 16, 2004

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Duplication at any time via copier, telephone, e-mail, World Wide Web or media of any type is a violation of the competition rules.

Correspondence about the problems and solutions should be addressed to:
Ms. Bonnie Leitch, AMC 8 Chair / bleitch@earthlink.net
548 Hill Avenue, New Braunfels, TX 78130

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:
Attn: Publications
American Mathematics Competitions
University of Nebraska-Lincoln
P.O. Box 81606
Lincoln, NE 68501-1606

1. **(B)** If 12 centimeters represents 72 kilometers, then 1 centimeter represents 6 kilometers. So 17 centimeters represents $17 \times 6 = 102$ kilometers.

2. **(B)** To form a four-digit number using 2, 0, 0 and 4, the digit in the thousands place must be 2 or 4. There are three places available for the remaining nonzero digit, whether it is 4 or 2. So the final answer is 6.

   OR

   Make a list: 2004, 2040, 2400, 4002, 4020 and 4200. So 6 numbers are possible.

3. **(A)** If 12 people order $\frac{18}{12} = 1\frac{1}{2}$ times too much food, they should have ordered $\frac{12}{\frac{3}{2}} = \frac{2}{3} \times 12 = 8$ meals.

   OR

   Let x be the number of meals they should have ordered. Then,

   $$\frac{12}{18} = \frac{x}{12},$$

   so

   $$x = 8.$$

4. **(B)** When three players start, one is the alternate. Because any of the four players might be the alternate, there are four ways to select a starting team: Lance-Sally-Joy, Lance-Sally-Fred, Lance-Joy-Fred and Sally-Joy-Fred.

5. **(D)** It takes 15 games to eliminate 15 teams.

6. **(C)** If Sally makes 55% of her 20 shots, she makes $0.55 \times 20 = 11$ shots. If Sally makes 56% of her 25 shots, she makes $0.56 \times 25 = 14$ shots. So she makes $14 - 11 = 3$ of the last 5 shots.

7. **(B)** A 26-year-old's target heart rate is $0.8(220 - 26) = 155.2$ beats per minute. The nearest whole number is 155.

8. **(B)** There are 7 two-digit numbers whose digits sum to 7: 16, 61, 25, 52, 34, 43 and 70.

9. **(D)** The sum of all five numbers is $5 \times 54 = 270$. The sum of the first two numbers is $2 \times 48 = 96$, so the sum of the last three numbers is $270 - 96 = 174$. The average of the last three numbers is $\frac{174}{3} = 58$.

10. **(E)** Aaron worked 75 minutes on Monday, 50 on Tuesday, 145 on Wednesday and 30 on Friday, for a total of 300 minutes or 5 hours. He earned $5 \times \$3 = \$15$.

11. **(C)** The largest, smallest and median occupy the three middle places, so the other two numbers, 9 and 4, are in the first and last places. The average of 9 and 4 is $\frac{9+4}{2} = 6.5$.

12. **(B)** The phone has been used for 60 minutes, or 1 hour, to talk, during which time it has used $\frac{1}{3}$ of the battery. In addition, the phone has been on for 8 hours without talking, which used an additional $\frac{8}{24}$ or $\frac{1}{3}$ of the battery. Consequently, $\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$ of the battery has been used, meaning that $\frac{1}{3}$ of the battery, or $\frac{1}{3}\times 24=8$ hours remain if Niki does not talk on her phone.

OR

Niki's battery has 24 hours of potential battery life. By talking for one hour, she uses $\frac{1}{3}\times 24=8$ hours of battery life. In addition, the phone is left on and unused for 8 hours, using an additional 8 hours. This leaves $24-8-8=8$ hours of battery life if the phone is on and unused.

13. **(E)** Bill is not the oldest, because if he were, the first two statements would be true. Celine is not the oldest, because if she were, the last two statements would be true. Therefore, Amy is the oldest. So the first two statements are false. The last statement must be true. This means that Celine is not the youngest, so Bill is the youngest. The correct order from oldest to youngest is Amy, Celine, Bill.

OR

The possible cases for the three statements are

	I	II	III
1	T	F	F
2	F	T	F
3	F	F	T

Case 1 leads to a contradiction of statements I and II.

Case 2 means Celine is youngest and neither Bill nor Amy is oldest, but one must be oldest.

Only case 3 is possible. The correct order from oldest to youngest is Amy, Celine, Bill.

14. **(C)** To find the area, subtract the areas of regions A, B, C, D and E from that of the surrounding square.

Square: $10\times 10=100$

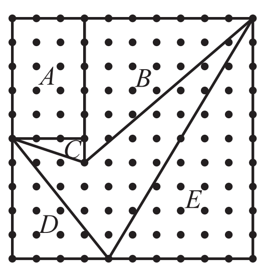
Region A: $3\times 5=15$

Region B: $\frac{1}{2}\times 6\times 7=21$

Region C: $\frac{1}{2}\times 1\times 3=1\frac{1}{2}$

Region D: $\frac{1}{2}\times 4\times 5=10$

Region E: $\frac{1}{2}\times 6\times 10=30$



The area is $100-\left(15+21+1\frac{1}{2}+10+30\right)=100-77\frac{1}{2}=22\frac{1}{2}$ square units.

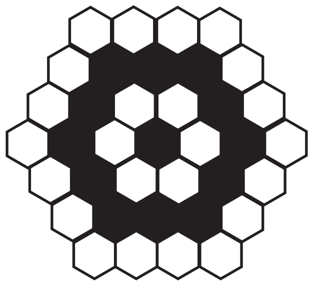
OR

By Pick's Theorem, the area of a polygon with vertices in a lattice is

$$(\text{number of points inside}) + \frac{\text{number of points on boundary}}{2} - 1.$$

In this case, the area is $21 + \frac{5}{2} - 1 = 22\frac{1}{2}$.

15. **(C)** The next border requires an additional $6 \times 3 = 18$ white tiles. A total of 24 white and 13 black tiles will be used, so the difference is $24 - 13 = 11$.



16. **(C)** Because the first pitcher was $\frac{1}{3}$ full of orange juice, after filling with water it contains 200 ml of juice and 400 ml of water. Because the second pitcher was $\frac{2}{5}$ full of orange juice, after filling it contains 240 ml of orange juice and 360 ml of water. In all, the amount of orange juice is 440 ml out of a total of 1200 ml or $\frac{440}{1200} = \frac{11}{30}$ of the mixture.

17. **(D)** The largest number of pencils that any friend can have is four. There are 3 ways that this can happen: $(4,1,1)$, $(1,4,1)$ and $(1,1,4)$. There are 6 ways one person can have 3 pencils: $(3,2,1)$, $(3,1,2)$, $(2,3,1)$, $(2,1,3)$, $(1,2,3)$ and $(1,3,2)$. There is only one way all three can have two pencils each: $(2,2,2)$. The total number of possibilities is $3+6+1 = 10$.

OR

The possible distributions of 6 pencils among 3 friends are the following:

1	1	4
1	2	3
1	3	2
1	4	1
2	1	3
2	2	2
2	3	1
3	1	2
3	2	1
4	1	1

The number of possible distributions is 10.

18. **(A)** Ben must hit 1 and 3. This means Cindy must hit 5 and 2, because she scores 7 using two different numbers, neither of which is 1 or 3. By similar reasoning, Alice hits 10 and 6, Dave hits 7 and 4, and Ellen hits 9 and 8. Alice hits the 6.

OR

Ellen's score can be obtained by either $10 + 7$ or $9 + 8$. In the first case, it is impossible for Alice to score 16. So Ellen's 17 is obtained by scoring 9 and 8, and Alice's total of 16 is the result of her hitting 10 and 6. The others scored $11 = 7 + 4$, $7 = 5 + 2$ and $4 = 3 + 1$.

19. **(B)** The numbers that leave a remainder of 2 when divided by 4 and 5 are 22, 42, 62 and so on. Checking these numbers for a remainder of 2 when divided by both 3 and 6 yields 62 as the smallest.

OR

The smallest whole number that is evenly divided by each of 3, 4, 5 and 6 is $\mathsf{LCM}\{3, 4, 5, 6\} = 2^2 \times 3 \times 5 = 60$. So the smallest whole number greater than 2 that leaves a remainder of 2 when divided by each of 3, 4, 5 and 6 is 62.

20. **(D)** Because the 6 empty chairs are $\frac{1}{4}$ of the chairs in the room, there are $6 \times 4 = 24$ chairs in all. The number of seated people is $\left(\frac{3}{4}\right)24 = 18$, and this is $\frac{2}{3}$ of the people present. It follows that

$$\frac{18}{\text{people present}} = \frac{2}{3}.$$

So there are 27 people in the room.

21. **(D)** In eight of the twelve outcomes the product is even: 1×2, 2×1, 2×2, 2×3, 3×2, 4×1, 4×2, 4×3. In four of the twelve, the product is odd: 1×1, 1×3, 3×1, 3×3. So the probability that the product is even is $\frac{8}{12}$ or $\frac{2}{3}$.

OR

To get an odd product, the result of both spins must be odd. The probability of odd is $\frac{1}{2}$ on Spinner A and $\frac{2}{3}$ on Spinner B. So the probability of an odd product is $\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{3}$. The probability of an even product, then, is $1 - \frac{1}{3} = \frac{2}{3}$.

22. **(B)** Because $\frac{2}{5}$ of all the women in the room are single, there are two single women for every three married women in the room. There are also two single women for every three married men in the room. So out of every $2 + 3 + 3 = 8$ people, 3 are men. The fraction of the people who are married men is $\frac{3}{8}$.

23. **(D)** The distance increases as Tess moves from J to K, and continues at perhaps a different rate as she moves from K to L. The greatest distance from home will occur at L. The distance decreases as she runs from L to M and continues at perhaps a different rate as she moves from M to J. Graph D shows these changes.

24. **(C)** By the Pythagorean Theorem, $HE = 5$. Rectangle $ABCD$ has area $10 \times 8 = 80$, and the corner triangles have areas $\frac{1}{2} \times 3 \times 4 = 6$ and $\frac{1}{2} \times 6 \times 5 = 15$. So the area of $EFGH$ is $80 - (2)(6) - (2)(15) = 38$. Because the area of $EFGH$ is $EH \times d$ and $EH = 5$, $38 = 5 \times d$, so $d = 7.6$.

25. **(D)** The overlap of the two squares is a smaller square with side length 2, so the area of the region covered by the squares is $2(4 \times 4) - (2 \times 2) = 32 - 4 = 28$. The diameter of the circle has length $\sqrt{2^2 + 2^2} = \sqrt{8}$, the length of the diagonal of the smaller square. The shaded area created by removing the circle from the squares is $28 - \pi\left(\frac{\sqrt{8}}{2}\right)^2 = 28 - 2\pi$.

The

American Mathematics Contest 8 (AMC 8)

Sponsored by

The Mathematical Association of America
University of Nebraska – Lincoln

Contributors

Akamai Foundation
American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour & Company
Mu Alpha Theta
National Council of Teachers of Mathematics
Pedagoguery Software, Inc.
Pi Mu Epsilon
Society of Actuaries
U. S. A. Math Talent Search
W. H. Freeman & Company