American Mathematics Competitions

16th Annual

AMC 8

Solutions Pamphlet

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This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Duplication at any time via copier, telephone, e-mail, World Wide Web or media of any type is a violation of the copyright law.

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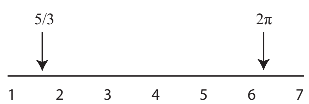
1. **Answer (B):** Brianna is half as old as Aunt Anna, so Brianna is 21 years old. Caitlin is 5 years younger than Brianna, so Caitlin is 16 years old.

2. **Answer (A):** The number 0 has no reciprocal, and 1 and -1 are their own reciprocals. This leaves only 2 and -2. The reciprocal of 2 is $\frac{1}{2}$, but 2 is not less than $\frac{1}{2}$. The reciprocal of -2 is $-\frac{1}{2}$, and -2 is less than $-\frac{1}{2}$.

3. **Answer (D):** The smallest whole number in the interval is 2 because $\frac{5}{3}$ is more than 1 but less than 2. The largest whole number in the interval is 6 because 2π is more than 6 but less than 7. There are five whole numbers in the interval. They are 2, 3, 4, 5, and 6.



4. **Answer (E):** The data are 1960(5%), 1970(8%), 1980(15%), and 1990(30%). Only graph (E) has these entries.

5. **Answer (C):** If the first year of the 8-year period was the final year of a principal's term, then in the next six years two more principals would serve, and the last year of the period would be the first year of the fourth principal's term. Therefore, the maximum number of principals who can serve during an 8-year period is 4.

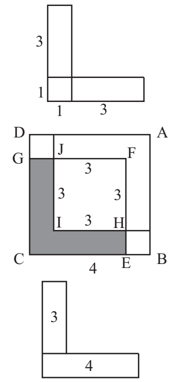
6. **Answer (A):** The L-shaped region is made up of two rectangles with area $3 \times 1 = 3$ plus the corner square with area $1 \times 1 = 1$, so the area of the L-shaped figure is $2 \times 3 + 1 = 7$.

   OR

   Square $FECG-$ square $FHIJ = 4 \times 4 - 3 \times 3 = 16 - 9 = 7$.

   OR

   The L-shaped region can be decomposed into a 4×1 rectangle and a 3×1 rectangle. So the total area is 7.



7. **Answer (B):** The only way to get a negative product using three numbers is to multiply one negative number and two positives or three negatives. Only two reasonable choices exist: $(-8) \times (-6) \times (-4) = (-8) \times (24) = -192$ and $(-8) \times 5 \times 7 = (-8) \times 35 = -280$. The latter is smaller.

8. **Answer (D):** The numbers on one die total $1+2+3+4+5+6 = 21$, so the numbers on the three dice total 63. Numbers 1, 1, 2, 3, 4, 5, 6 are visible, and these total 22. This leaves $63 - 22 = 41$ not seen.

9. **Answer (D):** The 3-digit powers of 5 are 125 and 625, so space 2 is filled with a 2. The only 3-digit power of 2 beginning with 2 is 256, so the outlined block is filled with a 6.

10. **Answer (E):** Shea is 60 inches tall. This is 1.2 times the common starting height, so the starting height was $\frac{60}{1.2} = 50$ inches. Shea has grown $60 - 50 = 10$ inches. Therefore, Ara grew 5 inches and is now 55 inches tall.

11. **Answer (C):** Twelve numbers ending with 1, 2, or 5 have this property. They are 11, 12, 15, 21, 22, 25, 31, 32, 35, 41, 42, and 35. In addition, we have 33, 24, 44, 36, and 48, for a total of 17. (Note that 20, 30, and 40 are not divisible by 0, since division by 0 is not defined.)

12. **Answer (D):** If the vertical joins were not staggered, the wall could be build with $\frac{1}{2}(100 \times 7) = 350$ of the two-foot blocks. To stagger the joins, we need only to replace, in every other row, one of the longer blocks by two shorter ones, placing one at each end. To minimize the number of blocks this should be done in rows 2, 4, and 6. This adds 3 blocks to the 350, making a total of 353.

13. **Answer (C):** Since $\angle ACT = \angle ATC$ and $\angle CAT = 36°$, we have $2(\angle ATC) = 180° - 36° = 144°$ and $\angle ATC = \angle ACT = 72°$. Because $\overline{TR}$ bisects $\angle ATC$, $\angle CTR = \frac{1}{2}(72°) = 36°$. In triangle CRT, $\angle CRT = 180° - 36° - 72° = 72°$. Note that some texts use $\angle ACT$ to define the angle and $m\angle ACT$ to indicate its measure.

14. **Answer (D):** The units digit of a power of an integer is determined by the units digit of the integer; that is, the tens digit, hundreds digit, etc... of the integer have no effect on the units digit of the result. In this problem, the units digit of 19^{19} is the units digit of 9^{19}. Note that $9^1 = 9$ ends in 9, $9^2 = 81$ ends in 1, $9^3 = 729$ ends in 9, and, in general, the units digit of odd powers of 9 is 9, whereas the units digit of even powers of 9 is 1. Since both exponents are odd, the sum of their units digits is $9 + 9 = 18$, the units digit of which is 8.

15. **Answer (C):** We have

$$AB + BC + CD + DE + EF + FG + GA = 4 + 4 + 2 + 2 + 1 + 1 + 1 = 15$$

16. **Answer (C):** The perimeter is $1000 \div 10 = 100$, and this is two lengths and two widths. The length of the backyard is $1000 \div 25 = 40$. Since two lengths total 80, the two widths total 20, and the width is 10. The area is $10 \times 40 = 400$.

17. **Answer (A):** We have

$$(1 \otimes 2) \otimes 3 = \frac{1^2}{2} \otimes 3 = \frac{1}{2} \otimes 3 = \frac{(\frac{1}{2})^2}{3} = \frac{\frac{1}{4}}{3} = \frac{1}{12},$$

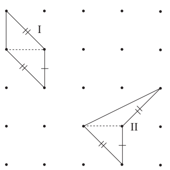
and

$$1 \otimes (2 \otimes 3) = 1 \otimes (\frac{2^2}{3}) = 1 \otimes \frac{4}{3} = \frac{1^2}{\frac{4}{3}} = \frac{3}{4}.$$

Therefore,

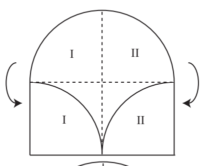
$$(1 \otimes 2) \otimes 3 - 1 \otimes (2 \otimes 3) = \frac{1}{12} - \frac{3}{4} = \frac{1}{12} - \frac{9}{12} = -\frac{8}{12} = -\frac{2}{3}.$$

18. **Answer (E):** Divide each quadrilateral as shown. The resulting triangles each have base 1, altitude 1, and area $\frac{1}{2}$, so the quadrilaterals each have area 1.



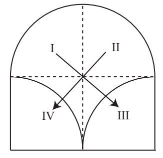
Three sides of quadrilateral I match those of quadrilateral II as indicated by matching marks. The fourth side of quadrilateral I is less than the fourth side of quadrilateral II, hence its perimeter is less, and choice (E) is correct.

19. **Answer (C):** Divide the semicircle in half and rotate each half down to fill the space below the quarter-circles. The figure formed is a rectangle with dimensions 5 and 10. The area is 50.



OR

Slide I into III and II into IV as indicated by the arrows to create the 5×10 rectangle.



20. **Answer (A):** Since the total value is \$1.02, you must have either 2 or 7 pennies. It is impossible to have 7 pennies, since the two remaining coins cannot have a value of 95 cents. With 2 pennies the remaining 7 coins have a value of \$1.00. Either 2 or 3 of these must be quarters. If you have 2 quarters, the other 5 coins would be dimes, and you would have no nickels. The only possible solution is 3 quarters, 1 dime, 3 nickels and 2 pennies.

21. **Answer (B):** Make a complete list of equally likely outcomes:

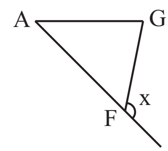
Keiko	Ephraim	Same Number of Heads?
H	HH	No
H	HT	Yes
H	TH	Yes
H	TT	No
T	HH	No
T	HT	No
T	TH	No
T	TT	Yes

The probability that they have the same number of heads is $\frac{3}{8}$.

22. **Answer (C):** The area of each face of the larger cube is $2^2 = 4$. There are six faces of the cube, so its surface area is $6(4) = 24$. When we add the smaller cube, we decrease the original surface area by 1, but we add $5(1^2) = 5$ units of area (1 unit for each of the five unglued faces of the smaller cube), This is a net increase of 4 from the original surface area, and 4 is $\frac{4}{24} = \frac{1}{6} \approx 16.7\%$ of 24. The closest value given is 17.

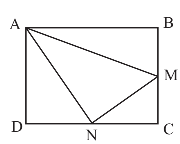
23. **Answer (B):** Since the average of all seven numbers is $6\frac{4}{7} = \frac{46}{7}$, the sum of the seven numbers is $7 \times \frac{46}{7} = 46$. The sum of the first four numbers is $4 \times 5 = 20$ and the sum of the last four numbers is $4 \times 8 = 32$. Since the fourth number is used in each of these two sums, the fourth number must be $(20 + 32) - 46 = 6$.

24. **Answer (D):** Since $\angle AFG = \angle AGF$ and $\angle GAF + \angle AFG + \angle AGF = 180°$, we have $20° + 2(\angle AFG) = 180°$. So $\angle AFG = 80°$. Also, $\angle AFG + \angle BFD = 190°$, so $\angle BFD = 100°$. The sum of the angles of $\triangle BFD$ is $180°$, so $\angle B + \angle D = 80°$.



Note: In $\triangle AFG, \angle AFG = \angle B + \angle D$. In general, an exterior angle of a triangle equals the sum of its remote interior angles. For example, in $\triangle GAF, \angle x = \angle GAF + \angle AGF$.
Note that, as in Problem 13, some texts use different symbols to represent an angle and its degree measure.

25. **Answer (B):** Three right triangles lie outside $\triangle AMN$. Their areas are $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{8}$ for a total of $\frac{5}{8}$ of the rectangle. The area of $\triangle AMN$ is $\frac{3}{8}(72) = 27$.



OR

Let the rectangle have sides of $2a$ and $2b$ so that $4ab = 72$ and $ab = 18$.Three right triangles lie outside triangle AMN, and their areas are $\frac{1}{2}(2a)(b)$, $\frac{1}{2}(2b)(a)$, $\frac{1}{2}(a)(b)$, for a total of $\frac{5}{2}(ab) = \frac{5}{2}(18) = 45$. The area of triangle AMN is $72 - 45 = 27$.